STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING TOEGEPASTE WISKUNDE

Report TW 88

Homogeneous Distributions

by

R. T. Seeley

April 1962

Introduction

In this article the nature of an arbitrary distribution f, homogeneous of degree λ for a complex λ, is described in terms of an expansion $f=\sum b_{mn} r^{\lambda} S_{mn}$ in spherical harmonics, and the Fourier transform is shown to have the form $f\hat{}=\sum b_{mn} \gamma_m(\lambda) r^{\lambda} S_{mn}$. The form of these expansions is slightly different for certain integer values of λ. The expansion of singular integral operators in spherical harmonics as in [2] together with the discussion of homogeneous distributions in [3], form the background of this investigation.

We consider distributions on real ν-dimensional space R_ν. Points in R_ν are denoted by $x=(x_1,\dots,x_\nu)$, and $|x|^2=\sum_1^{\nu} x_j^2$. The spherical coordinates (r,ω) of x are determined by $r=|x|$, $x=r\omega$. The unit sphere in R_ν is denoted by Ω.

Several spaces of test functions on R_ν appear, namely $D_K \subset D \subset S$, all consisting of infinitely differentiable functions. Those in D_K vanish for $|x| \geqq K$; those in S have $p(x)q(\partial/\partial x_1,\dots,\partial/\partial x_\nu)\varphi$ bounded for each polynomial p and q; and $D=\bigcup_{K=1}^{\infty} D_K$. In S (in D_K) there is a base of neighbourhoods of zero given by $U_n=\{\varphi;\varphi \text{ in } S \text{ (in } D_K),\ (1+|x|)^n\ |D^k\varphi(x)| < 1/n \text{ for } 0 \leqq k \leqq n\}$, where D^k runs over all differentiations $\partial^k/\partial x_1^{k_1}\dots\partial x_\nu^{k_\nu}$ of order k. D is usually not given a topology. At time the alternate notations $D(|x|<K)$, $D(R_\nu)$, and $S(R_\nu)$ are used.

The spaces D_K' and S' of distributions are respectively the continuous linear functionals on D_K and S; and $D'=\bigcap_{K=1}^{\infty} D_K'$.

Thus $S' \subset D' \subset D_K'$. Sometimes the notations $D'(R_\nu)$ and $S'(R_\nu)$ are used. The value of the distribution f on the test function φ is $\langle f,\varphi\rangle$.

$D(\Omega)$ is the space of C^∞ functions on the unit sphere Ω, with a base of neighbourhoods of zero given by $U_n=\{\psi;|D^k\psi(x/|x|) < 1/n, \text{ for } 0 \leqq k \leqq n \text{ and } |x| \geqq 1\}$. $D'(\Omega)$

is then the space of continuous functionals on $D(\Omega)$.

For φ in S, φ_t is defined by $\varphi_t(x)=\varphi(t\,x)$. Since for a continuous function f homogeneous of degree λ, with $\mathrm{Re}(\lambda)>-\nu$, we have $\int f(x)\,\varphi(x)dx=t^{\lambda+\nu}\int f(x)\varphi(t\,x)dx$, the following definition (given in [3]) is natural.

<u>Definition 1.</u> <u>The distribution f in $D'(R_\nu)$ is homogeneous of degree λ if and only if, for each $t>0$, $\langle f,\varphi\rangle=t^{\lambda+\nu}\langle f,\varphi_t\rangle$.</u>

The steps to the main theorem are as follows: § 1 obtains for $\mathrm{Re}(\lambda)>-\nu$ a representation $f=r^\lambda F$, where F is in $D'(\Omega)$; § 2 discusses the convergence of the expansion in spherical harmonics of a distribution F in $D'(\Omega)$; § 3 computes the Fourier transform of the individual terms in the expansion of f; § 4 combines these into the theorem, and makes a few applications.

§ 1 Here we establish Lemma 2, and the following corollary: if λ is any complex number, and f is in D' and homogeneous of degree λ, then f has an extension in S'; i.e. f is continuous on the larger space S.

<u>Definition 2.</u> <u>Let $\mathrm{Re}(\lambda)>-\nu$, and φ be in S. Then $P_\lambda\varphi$ is the function on Ω defined by $(P_\lambda\varphi)\,(\omega)=\int_0^\infty t^{\lambda+\nu-1}\,\varphi(t\,\omega)dt$.</u>

P_λ is continuous from S to $D(\Omega)$, since $\int_0^\infty t^{\lambda+\nu-1}D^n\varphi(tx/|x|)dt$ can be estimated in terms of the supremum of $(1+|x|)^m|D^k\varphi(x)|$ for $k\leqq n$ and sufficiently large m.

<u>Definition 3.</u> <u>If F is in $D'(\Omega)$, and $\mathrm{Re}(\lambda)>-\nu$, then $r^\lambda F$ is the distribution in $S'(R_\nu)$ defined by $\langle r^\lambda F,\varphi\rangle=\langle F,P_\lambda\varphi\rangle$.</u>

Since P_λ is continuous from S to $D(\Omega)$, the composition of F and P_λ is a continuous linear functional on S. Informally written, $\langle r^\lambda F,\varphi\rangle=\int_0^\infty\langle r^\lambda F(\omega),\varphi(r\,\omega)\rangle\,r^{-1}dr$.

<u>Definition 4.</u> <u>Let a(t) be a non-negative C^∞ function on R_1 with support in $1/2\leqq t\leqq 2$. Then for ψ in $D(\Omega)$, $A_\lambda\psi$ is defined by $(A_\lambda\psi)\,(x)=a(|x|)|x|^{-\lambda-\nu+1}\,\psi(x/|x|)/\int_0^\infty a(t)dt$.</u>

Thus A_λ depends on the arbitrarily chosen function a(t); but since we consider a fixed a(t) this dependence is not indicated in the notation. It is clear that A_λ is continuous from $D(\Omega)$ to $D(|x| < 2)$.

Lemma 1. Let $Re(\lambda) > -\nu$, and f in D' be homogeneous of degree λ. Then $\langle f,\varphi\rangle = \langle f, A_\lambda P\varphi\rangle$ for each φ in D.

Proof. The basic calculation is

$$1)\quad \left(\int_0^\infty a\right) \quad \langle f, A_\lambda P_\lambda \varphi\rangle = \langle f(x), |x| a(|x|) \int_0^\infty s^{\lambda+\nu-1} \varphi(sx)ds\rangle$$

$$= \int_0^\infty s^{\lambda+\nu-1} \langle f(x), [\varphi(x)a(|x|/s)|x|/s]_s \rangle ds$$

$$= \int_0^\infty s^{-2} \langle f(x), \varphi(x)a(|x|/s)|x| \rangle ds$$

$$= \int_0^\infty \langle f(x), \varphi(x)a(t|x|)|x| \rangle dt$$

$$= \langle f(x), \varphi(x) \int_0^\infty a(t|x|)d(t|x|)\rangle$$

$$= \left(\int_0^\infty a\right) \quad \langle f, \varphi\rangle .$$

The interchange of $\int$ and $\langle\ ,\ \rangle$ seems to be difficult to justify unless φ vanishes in a neighbourhood of the origin, so we first consider a φ with $\varphi=0$ for $|x| \leqq \epsilon \leqq 1/2$ and $|x| \geqq M \geqq 2$. Then the interchanges can be justified by showing that if ψ and ψ_1 are C^∞ functions vanishing for $|x| \leqq \epsilon$ and $|x| \geqq M$, and μ is any complex number then, in the topology of $D(|x| < M)$, $\psi_1(x) \int_0^A s^\mu \ (sx)ds \longrightarrow \psi_1(x) \int_0^\infty s^\mu \psi(sx)ds$ and $\psi_1(x)(A/N) \sum_{n=1}^N (An/N)^\mu \psi(Anx/N) \longrightarrow \int_0^A s^\mu \psi(sx)ds$. Since the derivatives of each of these expressions are linear combinations of the same type, it suffices to show that $\int_0^A s^\mu \psi(sx)ds \longrightarrow \int_0^\infty$ and $(A/N) \sum_1^N (An/N)^\mu \psi(Anx/N) \longrightarrow \int_0^A$ uniformly in $\epsilon \leqq |x| \leqq M$, for each μ.

For the first we have in $|x| \geqq \epsilon$ that $\left|\int_A^\infty s^\mu \psi(sx)ds\right|$

$$\leqq \sup_{t > 9x} (1+|x|t)^k |\psi(tx)| (1+t)^k (1+\epsilon t)^{-k} \int_A^\infty s^{Re(\mu)} (1+s)^{-k} ds ;$$

choosing $k > \mathrm{Re}(\mu)-1$ yields the result. For the convergence of the Riemann sums, we have

$$\left|\int_0^A -(A/N)\sum_1^N\right| \leq (A^2/N) \max_{\epsilon \leq |x| \leq M} |ds^{\mu}\psi(sx)/ds| ;$$

since $\psi(y)$ vanishes for $|y| \leq \epsilon$, we need only consider $\epsilon/M \leq s \leq A$, and $\max_{\substack{|x| \leq M \\ \epsilon/M \leq s \leq A}} |ds^{\mu}\psi(sx)/ds|$ is finite.

This justifies formula (1), and hence completes the lemma, for functions φ vanishing in a neighbourhood of the origin. Thus if g is defined by $\langle g,\varphi\rangle = \langle f, A_\lambda P_\lambda \varphi\rangle$, then f-g has support $\{x = 0\}$, and hence (see [4], p.99) $f-g = \sum_1^N L_n \delta$, where L_n is a homogeneous differential operator of order n with constant coefficients.

Now the n^{th} term of this sum is homogeneous of degree $-\nu-n$, and g is easily shown to be homogeneous of degree λ. Since distributions which are homogeneous of different degrees are linearly independent (see [3], p.86), we are led to f=g, and Lemma 1 is proved.

Now it is easy to define an F in $D'(\Omega)$ such that $f=r^\lambda F$, i.e. such that $\langle f,\varphi\rangle = \langle F, P_\lambda\varphi\rangle$: set $\langle F,\psi\rangle = \langle f, A_\lambda \psi\rangle$. Then $\langle f,\varphi\rangle = \langle f, A_\lambda P_\lambda \varphi\rangle = \langle F, P_\lambda \varphi\rangle$ as desired. In spite of the arbitrariness in A_λ, F is unique; for if ψ is in $D(\Omega)$ we have $\psi = P_\lambda A_\lambda \psi$, so that $r^\lambda G = f$ implies $\langle G,\psi\rangle = \langle G, P_\lambda A_\lambda \psi\rangle = \langle f, A_\lambda\psi\rangle = \langle F,\psi\rangle$ for each ψ.

Thus we have established

<u>Lemma 2.</u> <u>If $\mathrm{Re}(\lambda) > -\nu$, and f in $D'(R_\nu)$ is homogeneous of degree λ, then there is a unique F in $D'(\Omega)$ such that</u> $f=r^\lambda F$.

<u>Corollary.</u> <u>If f is in $D'(R_\nu)$, homogeneous of any complex degree λ, then f has an extension in S'</u>.

<u>Proof.</u> If $\mathrm{Re}(\lambda) > -\nu$, this follows from Lemma 1; for $A_\lambda P_\lambda$ is bounded from S to $D(|x|<2)$, so $\langle f,\varphi\rangle = \langle f, A_\lambda P_\lambda\varphi\rangle$ defines the extension. If $\mathrm{Re}(\lambda) \leq -\nu$, choose an integer k so that $2k+\mathrm{Re}(\lambda) > -\nu$. It is easy to check $|x|^{2k}$ f is homogeneous of degree $2k+\lambda$, hence continuous on S; and if $\chi(|x|)$ is a C^∞

cut-off function such that $\chi(|x|)=1$ for $|x|\leqq 1$, $\chi(|x|)=0$ for $|x|\geqq 2$, we have

$$2)\quad \langle f,\varphi\rangle = \langle f,\chi\varphi\rangle + \langle |x|^{2k} f, |x|^{-2k}(1-\chi)\varphi\rangle .$$

Here $\varphi\to\chi\varphi$ is continuous from S to $D(|x|<2)$, and $\varphi\to|x|^{-2k}(1-\chi)\varphi$ is continuous from S to S, so the right hand side of (2) is continuous on S.

§ 2 The-spherical harmonic expansion in $D'(\Omega)$.

Let S_m denote a real-valued normalized spherical harmonic of degree m; thus S_m is the restriction to Ω of a homogeneous harmonic polynomial of degree m, and $\int_\Omega |S_m|^2=1$. Let $\{S_{mn}\}$ denote an orthonormal basis for $L^2(\Omega)$ consisting of such spherical harmonics, S_{mn} being of degree m and n running from 1 to $(2m+\nu-2)(m+\nu-3)!/m!(\nu-2)!$ (see [1],p.237). If we define an operator L on $D(\Omega)$ by $L\psi=$ the restriction to Ω of $\Delta\psi(x/|x|)$, we have from [2] that

$$3)\qquad -m(m+\nu-2)\int_\Omega S_{mn}\psi = \int_\Omega S_{mn} L\psi .$$

The same reference shows that there are constants $C_{k,m}$ such that

$$4)\qquad D^k S_{mn}(x/|x|) \leqq C_{k,m}\, m^{k-1+\nu/2} \text{ in } |x|\geqq 1,$$

where D^k is an arbitrary differentiation of order k.

Each ψ in $D(\Omega)$ has an expansion $\psi=\sum a_{mn} S_{mn}$, with $a_{mn}=\int_\Omega S_{mn}\psi$. The estimate (3) shows that $a_{mn}=O(m^{-k})$ for every k. Taking into account the number of S_{mn} for each m, estimate (4) then shows that $\sum a_{mn} S_{mn}$ and all its derivatives converge uniformly in $|x|\geqq 1$, so that the series converges in $D(\Omega)$ to ψ.

To each F in $D'(\Omega)$ there corresponds a sequence of coefficients $b_{mn}=\langle F,S_{mn}\rangle$. If we set $F_M=\sum_{m\leqq M}\sum_n b_{mn} S_{mn}$, then F_M converges weakly to F:

$\langle F_M,\psi\rangle = \sum_{m\leqq M}\sum_n b_{mn}\, a_{mn} = \langle F,\psi\ \rangle \to \langle F,\psi\rangle$ for each ψ. Since

$\lim_{M\to\infty} \sum_{m\leq M} \sum_n b_{mn} a_{mn}$ exists for each $\{a_{mn}\}$ such that $a_{mn}=0(m^{-k})$ for all k, it follows that $b_{mn}=0(m^k)$ for some k.

We now have the expansion described in Theorem 1 below for the case $Re(\lambda)> -\nu$. Expend the F of Lemma 2 as $F=\sum\sum b_{mn} S_{mn}$. Then $\sum\sum b_{mn} r^{\lambda} S_{mn}$ converges weakly (in $S'(R_\nu)$) to f, since $\lim_{M\to\infty} \sum_{m=M} \sum_n b_{mn} \langle r^\lambda S_{mn},\varphi\rangle =$ $\lim \langle F_M, P_\lambda \varphi\rangle = \langle F, P_\lambda\varphi\rangle = \langle f,\varphi\rangle$.

§ 3 Fourier transforms.

For $Re(\lambda)> -\nu$, $r^\lambda S_m(\omega)$ is a locally integrable function on R with polynomial growth at ∞, hence defines a distribution on S, homogeneous of degree λ. Here we compute its Fourier transform $(r^\lambda S_m)^\wedge$, and consider the analytic extension to $Re(\lambda)\leq -\nu$. The method of calculation is borrowed from [2].

The Fourier transform of φ in S is the function $\varphi^\wedge$ defined by $\varphi^\wedge(y)=\int e^{ix\cdot y}\varphi(x)dx$; this is a continuous 1-1 transformation on S, whose inverse is given by $\varphi^\vee(x)=(2\pi)^{-\nu}\int e^{-ix\cdot y}\varphi(y)dy$. (See [5], p.105). The Fourier transform of f in S' is the distribution $f^\wedge$ in S' given by $\langle f^\wedge,\varphi\rangle = \langle f,\varphi^\wedge\rangle$. Thus $^\wedge$ and $^\vee$ are reciprocal isomorphisms on S'. One has, immediately, for an arbitrary polynomial P, that

5) $$[P(x)]^\wedge=(2\pi)^\nu P(-i\,\partial/\partial x_1,\dots,-i\,\partial/\partial x_\nu)\delta$$

and

6) $$[P(\partial/\partial x_1,\dots,\partial/\partial x_\nu)\delta]^\wedge=P(-ix_1,\dots,-ix_\nu).$$

It is easy to see that the distribution $(r^\lambda S_m)^\wedge$ corresponds to the function $(r^\lambda S_m)^\wedge(y)= \lim_{\epsilon\to 0}\int_{\epsilon\leq|x|\leq 1/\epsilon} |x|^\lambda e^{-ix\cdot y} S_m(x/|x|)dx$, for all values of λ such that this limit exists uniformly in each ball $|y|\leq K$. This turns out to include the strip $-\nu<Re(\lambda)<(1-\nu)/2$. The analytic expression obtained in this strip is then valid for all values of λ, by analytic continuation.

Consider thus $-\nu < \mathrm{Re}(\lambda) < (1-\nu)/2$, and set $x=r\omega$, $y=\rho\sigma$ $(|\omega|=|\sigma|=1)$; then $(r^{\lambda} S_m)^{\wedge}(\rho\sigma)=$

$$\lim_{\epsilon\to 0} \int_{\epsilon}^{1/\epsilon} r^{\lambda+\nu-1} \int_{\Omega} e^{i\rho r\sigma.\omega} S_m(\omega)d\Omega\, dr=$$

$$\lim \rho^{-\lambda-\nu} \int_{\epsilon}^{1/\epsilon} s^{\lambda+\nu-1} \int_{\Omega} e^{-is\sigma.\omega} S_m(\omega)d\Omega\, ds.$$

Further calculation depends on the formulas

7) $$e^{is\cos\varphi} = 2^a \Gamma(a)s^{-a} \sum_{k=0}^{\infty} (i)^k(k+a)\, J_{k+a}(s)C_k^a(\cos\varphi)$$

([1], p.213),

where $C_k^a(t)$ is a Gegenbauer polynomial;

8) $$\int_{\Omega} C_j^{a}(\sigma.\omega)S_m(\omega)d\Omega = \delta_{jm}\, S_m(\sigma)4\pi^{1+a}/(2m+\nu-2)\,\Gamma(a)$$

([1],p.247);

and

9) $$\int_0^{\infty} t^{\lambda+\nu/2}\, J_{m-1+\nu/2}(t)dt =$$

$$2^{\lambda+\nu/2}\,\Gamma((m+\nu+\lambda)/2)/\,\Gamma((m-\lambda)/2),$$

for $-m-\nu<\mathrm{Re}(\lambda)<(1-\nu)/2$. ([1], p.49).

Thus setting the letter a in formulas (7) and (8) equal to $(\nu/2)-1$, we obtain

10) $$\int_{\Omega} e^{-is\sigma.\omega} S_m(\omega)d\Omega = 2\pi^{\nu/2}(-i)^m(s/2)^{1-\nu/2}\, J_{m-1+\nu/2}(s)\, S_m(\sigma),$$

and

11) $$(r^{\lambda} S_m)^{\wedge}(\rho\sigma)=\rho^{-\lambda-\nu} S_m(\sigma)(-i)^m\pi^{/2}2^{\lambda+\nu}\Gamma((m+\nu+\lambda)/2)/\Gamma(m-\lambda)/2),$$

for $-\nu<\mathrm{Re}(\lambda)<(1-\nu)/2$. It follows easily that for the same values of λ

12) $$(r^{\lambda}S_m)^{\vee}(\rho\sigma)=\rho^{-\lambda-\nu}S_m(\sigma)(i)^m\pi^{-\nu/2}2^{\lambda}\Gamma((m+\nu+\lambda)/2)/\Gamma((m-\lambda)/2$$

Then, if $r^\lambda S_m$ is defined for all λ (except possible poles) as the analytic extension from $\mathrm{Re}(\lambda) > -\nu$, we have for all λ

$$13)\qquad [1/\Gamma((m+\nu+\lambda)/2)]\ (r^\lambda S_m)^\wedge = [(-i)^m\ \pi^{\nu/2}\ 2^{\lambda+\nu}/\Gamma((m-\lambda)/2)]\ r^{-\lambda-\nu} S_m.$$

Since for any λ either $\mathrm{Re}(\lambda) > -\nu$ or $\mathrm{Re}(-\lambda-\nu) > -\nu$, at least one of $r^\lambda S_m$ and $r^{-\lambda-\nu} S_m$ is always defined as a regular distribution. Formula (13) then defines the other of these as a Fourier transform or inverse Fourier Transform, except for the values of λ which yield a pole of the gamma functions occurring in (13). In this way <u>$r^\lambda S_m$ is defined by formula (13) as a distribution in $S'(R\ )$ except for $\lambda=-\nu-m-2k$, $k=0,1,2,\ldots$</u> The fact that this extension is possible can also be checked directly by a using a Taylor expansion of the test functions; this is done for the case m=0 in [3].

Since at least one side in (13) is always non-zero, the poles of the gamma function do not correspond to the zero distribution, but rather to distributions concentrated at the origin. In fact, if $\lambda=m+2k$ then $r^\lambda S_m$ is $r^{2k}H_m$, where H_m is a harmonic polynomial. Thus from (5) we have

$$14)\qquad (r^{m+2k}\ S_m)^\wedge = (2\pi)^\nu (-i)^{m+2k}\ \Delta^k H_m(\partial/\partial x_1,\ldots,\partial/\partial x_\nu)\delta$$

and

$$15)\qquad (r^{m+2k}\ S_m)^\vee = (i)^{m+2k}\ \Delta^k\ H_m(\partial/\partial x_1,\ldots,\partial/\partial x_\nu)\delta\ ,$$

where Δ is the Laplacian, and the δ is Dirac's.

Thus $r^\lambda S_m$, defined for $\mathrm{Re}(\lambda) > -\nu$ as a regular distribution, has an analytic extension to the whole complex λ-plane except for poles at $\lambda=-m-2k$. Its Fourier transform is given either by (13) or by (14), and the inverse transform by (13) or by (15).

§ 4 The expansion and transform of a homogeneous distribution

Theorem 1. Let f be a distribution in $D'(R_\nu)$, homogeneous of degree λ, and $r^\lambda S_{mn}$ be defined for $\lambda \neq -\nu, -\nu-1, \ldots$ by analytic continuation from $Re(\lambda) > -\nu$.

i) If λ is not an integer of the form $-\nu-k$ $(k=0,1,\ldots)$, then $f = \sum_m \sum_n b_{mn} r^\lambda S_{mn}$, where $b_{mn} = O(m^k)$ for some k, and the series is weakly convergent in $S'(R_\nu)$.

ii) If $\lambda = -\nu-N$ for some $N=0,1,\ldots$, then $f = P_N(\partial/\partial x_1, \ldots, \partial/\partial x_\nu)\delta + \sum_m^* \sum_n b_{mn} r^{-\nu-N} S_{mn}$, where P_N is a homogeneous polynomial of degree N, δ is the Dirac δ, and $\sum_m^*$ is the sum over all $m \geq 0$ such that $m \neq N-2k$ for a $k=0,1,2,\ldots$. The series converges as in (i).

iii) The Fourier transform of f is obtained by term-by-term application of (13), (14), or (6).

Proof. Part (iii) follows from the fact that the Fourier transform is continuous in the weak topology of distributions. Part (i) follows, for $Re(\lambda) > -\nu$, from Lemma 2 and the last paragraph of § 2. If $Re(\lambda) \leqq -\nu$, then a trivial check shows that $f^\wedge$ is homogeneous of degree $-\lambda-\nu$; and $Re(-\lambda-\nu) \geqq 0$, so $f^\wedge$ may be expanded as in (i). Applying the inverse Fourier transform term-by-term yields the expansion for f.

An immediate consequence is

Corollary 1. Any distribution homogeneous of degree λ, $\lambda \neq -\nu-N$, has the form $r^\lambda F$, where F is in $D'(\Omega)$. If $\lambda = -\nu-N$, then $f = P_N(\partial/\partial x)\delta + r^{-\nu-N} F$, where F is orthogonal to all S_m with $m=N, N-2, \ldots$.

We say that a distribution F in $D'(\Omega)$ corresponds to a distribution $f_\lambda = r^\lambda F$ in $S'(R_\nu)$ if and only if, for each φ vanishing in a neighbourhood of the origin, $\langle f, \varphi \rangle = \int_0^\infty r^{\lambda+\nu-1} \langle F, \varphi_r \rangle \, dr$. Thus if $\lambda = -\nu-N$, the f_λ above is not uniquely determined by F.

Corollary 2. For each F in $D'(\Omega)$, and for $\lambda \neq -\nu-N$, there is a corresponding unique distribution $f_\lambda = r^\lambda F$.

If $\lambda=-\nu-N$, there is a corresponding f_λ if and only if $\langle F,S_{mn}\rangle =0$ for $m=N,N-2,\ldots$.

Proof. Let $F=\sum b_{mn}\, S_{mn}$. If $\lambda\neq -\nu-N$, then f_λ is uniquely determined as $f_\lambda =\sum\sum b_{mn}\, r^\lambda\, S_{mn}$. If $\lambda= -\nu-N$, and $f_{-\nu-N}$ corresponds to F, we can expand $f_{-\nu-N}=$

$\sum^*\sum c_{mn}\, r^{-\nu-N}\, S_{mn}+P_N(\partial/\partial x)\delta$.

Applying $f_{-\nu-N}$ to $A_\lambda\, S_{mn}$ we find that $c_{mn}=b_{mn}$ for all m,n, and that b_{mn} vanishes for those m not occuring in $\sum^*$. The polynomial P_N is thus arbitrary, and the rest determined by F.

It is easy to show that, when it exists, $r^\lambda \circ F$ is the analytic extension of $r^\lambda F$ from $Re(\lambda) > -\nu$.

Applying Corollary 2 to regular (integrable) distributions in $D'(\Omega)$, we have

Corollary 3. If f is a function homogeneous of degree λ and locally integrable in $|x|\geqq 1$, and $\lambda\neq -\nu-N$, then f corresponds to a unique distribution homogeneous of degree λ . If $\lambda=-\nu-N$, then f corresponds to a distribution homogeneous of degree λ if and only if $\int_\Omega f(\omega)S_{mn}(\omega)d\Omega$ for all $m=N,N-2,\ldots \geqq 0$.

References

[1] A. Erdélyi et al., Higher Transcendental Functions, vol.2, New York, 1953.

[2] A.P. Calderon and A. Zygmund, "Singular integral operators and differential equations", American Journal of Mathematics, vol 79 (1957), pp. 901-921.

[3] I.M. Gelfand and G.E. Schilow, Verallgemeinerte Funktionen, vol 1, Berlin, 1960.

[4] L. Schwartz, Theorie des Distributions, vol 1, Paris, 1950.

[5] L. Schwartz, Theorie des Distributions, vol.2, Paris, 1951.